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Some q-series identities related to divisor functions☆

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Abstract

The generating functions of the divisor functions $\sigma_k(n) = \sum_{d|n} d^k$ are expressed as sums of products of the series

$$U_m(q) := \sum_{n=1}^{\infty} n^m q^n \prod_{j=n+1}^{\infty} (1 - q^j), \quad m = 1, \ldots, k+1,$$

and vice versa. Other related q-series identities are derived, including

$$\sum_{n=k}^{\infty} \binom{n}{k} q^n \prod_{j=n+1}^{\infty} (1 - q^j) = \sum_{j_1=1}^{\infty} \frac{q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{q^{j_k}}{1-q^{j_k}}.$$

1. Introduction

Let $\sigma_j(n)$ denote the sum of the jth powers of the divisors of n,

$$\sigma_j(n) := \sum_{d|n} d^j, \tag{1.1}$$

and for $q \in \mathbb{C}$, $|q| < 1$ define the function

$$S_j(q) := \sum_{n=1}^{\infty} \sigma_j(n) q^n. \tag{1.2}$$

It is well known that these generating functions $S_j(q)$ have the following expansions as Lambert series:

$$S_j(q) = \sum_{n=1}^{\infty} \frac{n^j q^n}{1-q^n}; \tag{1.3}$$

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this can be seen by expanding $q^n/(1-q^n)$ as a geometric series and collecting the coefficients of the powers of q.

The simplest case is $j=0$ when we have $\sigma_0(n)=d(n)$, the divisor function counting the number of divisors of n. It was shown by Uchimura [8] that

$$\sum_{n=1}^{\infty} d(n)q^n = \sum_{n=1}^{\infty} nq^n \prod_{j=n+1}^{\infty} (1-q^j). \tag{1.4}$$

In a later paper [10], Uchimura studied the more general expressions

$$U_k(q) := \sum_{n=1}^{\infty} n^k q^n \prod_{j=n+1}^{\infty} (1-q^j), \tag{1.5}$$

for arbitrary integers $k \geqslant 1$, and obtained the following result.

Theorem A. *For $k \geqslant 1$, we have*

$$U_k(q) = \sum \frac{k!}{n_1! \cdots n_k!} \left(\frac{S_0(q)}{1!}\right)^{n_1} \left(\frac{S_1(q)}{2!}\right)^{n_2} \cdots \left(\frac{S_k(q)}{k!}\right)^{n_k}, \tag{1.6}$$

where the summation extends over all integers $n_1, \ldots, n_k \geqslant 0$ with $n_1 + 2n_2 + \cdots + kn_k = k$.

We note that this result provides some interesting formulas for the divisor functions.

Corollary 1. *Let $c_k(n)$ denote the coefficient of q^n in $U_k(q)$. Then*

$$c_1(n) = d(n), \tag{1.7}$$

$$c_2(n) = \sigma_1(n) + \sum_{j=1}^{n-1} d(j)d(n-j), \tag{1.8}$$

$$c_3(n) = \sigma_2(n) + 3\sum_{j=1}^{n-1} d(j)\sigma_1(n-j) + \sum_{\substack{j+k+l=n \\ j,k,l \geqslant 1}} d(j)d(k)d(l), \tag{1.9}$$

$$\begin{aligned} c_4(n) = \sigma_3(n) &+ 3\sum_{j=1}^{n-1} \sigma_1(j)\sigma_1(n-j) + 4\sum_{j=1}^{n-1} d(j)\sigma_2(n-j) \\ &+ 6\sum_{\substack{j+k+l=n \\ j,k,l \geqslant 1}} d(j)d(k)\sigma_1(l) + \sum_{\substack{j_1+\cdots+j_k \\ j_1,\ldots,j_4 \geqslant 1}} d(j_1)\cdots d(j_4). \end{aligned} \tag{1.10}$$

Proof. From (1.6) we obtain

$$U_1(q) = S_0(q), \tag{1.11}$$

$$U_2(q) = S_1(q) + S_0(q)^2, \tag{1.12}$$

$$U_3(q) = S_2(q) + 3S_0(q)S_1(q) + S_0(q)^3, \tag{1.13}$$

$$U_4(q) = S_3(q) + 3S_1(q)^2 + 4S_0(q)S_2(q) + 6S_0(q)^2S_1(q) + S_0(q)^4. \tag{1.14}$$

The results now follow by multiplying the appropriate series (1.2) together, and equating the coefficients of q^n. (Note that (1.7) is just Uchimura's result (1.4).) $\square$

It is the purpose of this paper to study further the sums $S_j(q)$ and $U_j(q)$ and their connections with one another and with certain q-series. In Section 2, we express the sums $S_k(q)$ in terms of the $U_j(q)$. These $U_j(q)$ are closely related to certain recurrence sequences of polynomials; this will be shown in Section 3. Certain q-series identities are derived in Sections 4 and 5, and a class of binomial sums is evaluated in Section 6.

2. Expressions for the $S_j(q)$

It may be of interest to have expressions for the divisor functions $\sigma_j(n)$ alone, as opposed to expressions of the type (1.7)–(1.10). This can be achieved by writing the generating functions $S_j(q)$ in terms of $U_j(q)$. From (1.11) to (1.14) we obtain immediately, in addition to (1.11) itself,

$$S_1(q) = U_2(q) - U_1(q)^2, \tag{2.1}$$

$$S_2(q) = U_3(q) - 3U_1(q)U_2(q) + 2U_1(q)^3, \tag{2.2}$$

$$S_3(q) = U_4(q) - 4U_1(q)U_3(q) - 3U_2(q)^2 + 12U_1(q)^2U_2(q) - 6U_1(q)^4. \tag{2.3}$$

In general, we have the following result.

Theorem 1. *For $k \geqslant 1$, we have*

$$S_{k-1}(q) = \sum \frac{(-1)^{n_1+\cdots+n_k-1}}{n_1!\cdots n_k!}(n_1+\cdots+n_k-1)!\left(\frac{U_1(q)}{1!}\right)^{n_1}\cdots\left(\frac{U_k(q)}{k!}\right)^{n_k}, \tag{2.4}$$

where the summation extends over all integers $n_1,\ldots,n_k \geqslant 0$ with $n_1 + 2n_2 + \cdots + kn_k = k$.

Proof. We use the (exponential) complete Bell polynomials $Y_k = Y_k(x_1,\ldots,x_k)$ defined by

$$\exp\left\{\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right\} = 1 + \sum_{k=1}^{\infty} Y_k \frac{t^k}{k!}.$$

(see, e.g., [4, p. 134] or [1, p. 205]) and note that $x_j = S_{j-1}(q)$ and $Y_k = U_k(q)$; see also [10]. After some renaming, we have

$$\sum_{k=1}^{\infty} S_{k-1}(q)\frac{t^k}{k!} = \log\left\{1 + \sum_{j=1}^{\infty} U_j(q)\frac{t^j}{j!}\right\}.$$

We can now see that

$$S_{k-1}(q) = L_k = L_k(U_1,\ldots,U_k),$$

where L_k is the logarithmic polynomial (see, e.g., [4, p. 140]) which can be expressed as

$$L_k = \sum_{m=1}^{k} (-1)^{m-1}(m-1)!\, B_{k,m}. \tag{2.5}$$

The polynomials $B_{k,m}$ are the partial Bell polynomials, expressible as

$$B_{k,m} = \sum \frac{k!}{n_1! \cdots n_k!} \left(\frac{U_1}{1!}\right)^{n_1} \left(\frac{U_2}{2!}\right)^{n_2} \cdots \left(\frac{U_k}{k!}\right)^{n_k},$$

where the summation extends over all integers $n_1, \dots, n_k \geqslant 0$ with $n_1 + 2n_2 + \cdots + kn_k = k$ and $n_1 + n_2 + \cdots + n_k = m$. The identity (2.4) now follows directly from (2.5) and (2.6). $\square$

We note that (1.11) and (2.1)–(2.3) are the first four special cases of (2.4).

3. A polynomial sequence

Following Uchimura [8] in the case $k = 1$, we define for a fixed $k \geqslant 1$ the sequence of polynomials $U_{k,m}(q)$, $m = 1, 2, \dots,$ recursively by

$$\begin{aligned} U_{k,1}(q) &= q, \\ U_{k,m}(q) &= m^k q^m + (1 - q^m) U_{k,m-1}(q) \quad (m \geqslant 2). \end{aligned} \tag{3.1}$$

Thus,

$$\begin{aligned} U_{k,2}(q) &= q + 2^k q^2 - q^3, \\ U_{k,3}(q) &= q + 2^k q^2 + (3^k - 1)q^3 - q^4 - 2^k q^5 + q^6, \\ U_{k,4}(q) &= q + 2^k q^2 + (3^k - 1)q^3 + (4^k - 1)q^4 - (2^k + 1)q^5 \\ &\quad - (2^k - 1)q^6 - (3^k - 1)q^7 + q^8 + 2^k q^9 - q^{10}. \end{aligned}$$

These polynomials have degree $m(m+1)/2$, and by induction we see that

$$\begin{aligned} U_{k,m}(q) &= q(1 - q^2)(1 - q^3)\cdots(1 - q^m) + 2^k q^2 (1 - q^3)\cdots(1 - q^m) \\ &\quad + 3^k q^3 (1 - q^4)\cdots(1 - q^m) + \cdots + m^k q^m. \end{aligned}$$

It is now clear that the coefficients of q^n, with $n \leqslant m$, in $U_{k,m}(q)$ are the same as those in $U_k(q)$. Hence we have shown the following result (see also [8, Theorem 1]).

Theorem 2. *Let $c_k(n)$ be defined as in Corollary 1, and let $a_n^{(k,m)}$ be the coefficient of q^n in $U_{k,m}(q)$. Then for any $n \leqslant m$ we have $c_k(n) = a_n^{(k,m)}$.*

Using (3.1), we can construct the following table:

n	$c_k(n)$	n	$c_k(n)$
1	1	6	$6^k - 2^k$
2	2^k	7	$7^k - 3^k - 2^k$
3	$3^k - 1$	8	$8^k - 3^k - 2^k + 1$
4	$4^k - 1$	9	$9^k - 4^k - 3^k + 1$
5	$5^k - 2^k - 1$	10	$10^k - 4^k - 3^k + 1$

We note that the asymptotic behaviour of the recursive sequences similar to the ones defined in (3.1) has been studied in [2].

4. A q-series identity

In [2] it was shown that for each $k \geqslant 1$ there exists a polynomial $M_k(x_1, \ldots, x_k)$ with rational coefficients such that

$$\sum_{m=1}^{\infty} \frac{(-1)^m q^{\binom{m+1}{2}}}{(q)_m (1-q^m)^k} = M_k(S_0(q), S_1(q), \ldots, S_{k-1}(q)). \tag{4.1}$$

In this section we will derive this formula from Theorem A, giving a shorter and more direct proof. For $k \geqslant 1$, denote

$$V_k(q) := \sum_{n=k}^{\infty} \binom{n}{k} q^n \prod_{j=n+1}^{\infty} (1-q^j). \tag{4.2}$$

Lemma 1. *For $k \geqslant 1$, we have*

$$V_k(q) = q^{-\binom{k}{2}} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} q^{\binom{m+k}{2}}}{(q)_m (1-q^m)^k}. \tag{4.3}$$

Proof. Using the identities

$$\prod_{j=n+1}^{\infty} (1-q^j) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} q^{(m-1)n} q^{\binom{m}{2}}}{(q)_{m-1}}$$

(see [8, p. 133]) and

$$\sum_{n=k}^{\infty} \binom{n}{k} t^n = \frac{t^k}{(1-t)^{k+1}} \quad (|t| < 1),$$

we get with (4.2),

$$V_k(q)=\sum_{n=k}^{\infty}\sum_{m=1}^{\infty}\binom{n}{k}\frac{(-1)^{m-1}q^{(m-1)n}q^nq^{\binom{n}{2}}}{(q)_{m-1}}=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}q^{\binom{m}{2}}}{(q)_{m-1}}\sum_{n=k}^{\infty}\binom{n}{k}q^{mn},$$

i.e.,

$$V_k(q)=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}q^{\binom{m}{2}}q^{km}}{(q)_{m-1}(1-q^m)^{k+1}}. \tag{4.4}$$

This proves (4.3) if we note that

$$(1-q^m)(q)_{m-1}=(q)_m \quad\text{and}\quad \binom{m}{2}+km=\binom{m+k}{2}-\binom{k}{2}. \qquad \square$$

Theorem 3. *For $k\geqslant 1$, we have*

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}q^{\binom{m+1}{2}}}{(q)_m(1-q^m)^k}=\sum_{i=1}^{k}\left\{\sum_{j=0}^{k-i}\binom{k-1}{j+i-1}\frac{s(j+i,i)}{(j+i)!}\right\}U_i(q), \tag{4.5}$$

where the $s(m,n)$ are the Stirling numbers of the first kind, and $U_i(q)$ as in Section 1.

Proof. Using the binomial theorem, it is easy to verify that

$$\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{q^{jm}}{(1-q^m)^j}=\frac{q^m}{(1-q^m)^k};$$

hence we get with (4.4),

$$\sum_{j=1}^{k}\binom{k-1}{j-1}V_j(q)=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}q^{\binom{m+1}{2}}}{(q)_m(1-q^m)^k}. \tag{4.6}$$

On the other hand, from a well-known relation for the Stirling numbers of the first kind (see, e.g., [4, p. 213]) we obtain

$$\binom{n}{k}=\sum_{j=1}^{k}\frac{s(k,j)}{k!}n^j, \tag{4.7}$$

and therefore with (4.2) and (1.5),

$$V_k(q)=\sum_{j=1}^{k}\frac{s(k,j)}{k!}U_j(q).$$

With (4.6) we now get

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}q^{\binom{m+1}{2}}}{(q)_m(1-q^m)^k}=\sum_{j=1}^{k}\binom{k-1}{j-1}\sum_{i=1}^{j}\frac{s(j,i)}{j!}U_i(q). \tag{4.8}$$

Upon changing the order of summation on the right-hand side, we obtain (4.5). $\square$

Remarks. (a) It is now clear that Theorems A and 3 together prove (4.1) and provide explicit expressions for the polynomials M_k.

(b) The right-hand side of (4.5) can be written as

$$\sum_{i=1}^{k} a(k,i)U_i(q),$$

where the coefficients $a(k,i)$ can be determined by

$$a(1,1)=1, \qquad a(k+1,i+1)=\frac{k}{k+1}a(k,i+1)+\frac{1}{k+1}a(k,i). \tag{4.9}$$

This is easy to verify: first we see that

$$a(1,1)=\frac{s(1,1)}{1!}=1.$$

For the recursion in (4.9), we use the well-known recursion for the Stirling numbers of the first kind (see, e.g., [4, p. 214]):

$$s(j+i,i)=s(j+i+1,i+1)+(j+i)s(j+i,i+1).$$

Then

$$\begin{aligned}
&\frac{k}{k+1}a(k,i+1)+\frac{1}{k+1}a(k,i)\\
&=\frac{1}{k+1}\left\{k\sum_{j=0}^{k-i-1}\binom{k-1}{j+i}\frac{s(j+i+1,i+1)}{(j+i+1)!}+\sum_{j=0}^{k-i}\binom{k-1}{j+i-1}\frac{s(j+i,i)}{(j+i)!}\right\}\\
&=\frac{1}{k+1}\left\{\sum_{j=0}^{k-i-1}\binom{k}{j+i+1}\frac{s(j+i+1,i+1)}{(j+i)!}\right.\\
&\quad\left.+\sum_{j=0}^{k-i}\binom{k-1}{j+i-1}\frac{s(j+i+1,i+1)+(j+i)s(j+i,i+1)}{(j+i)!}\right\}\\
&=\frac{1}{k+1}\left\{\sum_{j=0}^{k-i}\left[\binom{k}{j+i+1}+\binom{k-1}{j+i-1}\right]\frac{s(j+i+1,i+1)}{(j+i)!}\right.\\
&\quad\left.+\sum_{j=0}^{k-i-1}\binom{k-1}{j+i}\frac{s(j+i+1,i+1)}{(j+i)!}\right\}\\
&=\frac{1}{k+1}\sum_{j=0}^{k-i}\binom{k+1}{j+i+1}\frac{s(j+i+1,i+1)}{(j+i)!}\\
&=\sum_{j=0}^{k-i}\binom{k}{j+i}\frac{s(j+i+1,i+1)}{(j+i+1)!}=a(k+1,i+1).
\end{aligned}$$

This proves (4.9). We can use (4.9) to prove by induction that

$$a(k,1)=\frac{1}{k},\qquad a(k,k)=\frac{1}{k!}$$

for all $k \geqslant 1$. It is also easy to obtain the following table.

$k\backslash i$	1	2	3	4	5	6
1	1					
2	$\frac{1}{2}$	$\frac{1}{2}$				
3	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$			
4	$\frac{1}{4}$	$\frac{11}{24}$	$\frac{1}{4}$	$\frac{1}{24}$		
5	$\frac{1}{5}$	$\frac{5}{12}$	$\frac{7}{24}$	$\frac{1}{12}$	$\frac{1}{120}$	
6	$\frac{1}{6}$	$\frac{137}{360}$	$\frac{5}{16}$	$\frac{17}{144}$	$\frac{1}{48}$	$\frac{1}{720}$

This leads us to conjecture the identity

$$\sum_{i=1}^{k} a(k,i)=1 \tag{4.10}$$

for all $k \geqslant 1$, which is easy to verify by going from (4.5) back to (4.8) and using (4.7):

$$\sum_{i=1}^{k} a(k,i)=\sum_{j=1}^{k}\binom{k-1}{j-1}\sum_{i=1}^{j}\frac{s(j,i)}{j!}=\sum_{j=1}^{k}\binom{k-1}{j-1}\binom{1}{j}=\binom{k-1}{0}=1.$$

5. A multiple series representation

One of the main results in [8] is the identity

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}q^{\binom{m+1}{2}}}{(q)_m(1-q^m)}=\sum_{m=1}^{\infty}\frac{q^m}{1-q^m}. \tag{5.1}$$

(In fact, this identity was known much earlier; see [7].) The following finite analogue of (5.1) was given in [6]:

$$\sum_{m=1}^{n}\frac{(-1)^{m-1}q^{\binom{m+1}{2}}}{1-q^m}\begin{bmatrix} n\\ m\end{bmatrix}=\sum_{m=1}^{n}\frac{q^m}{1-q^m}, \tag{5.2}$$

where $\left[\begin{smallmatrix} n\\ m\end{smallmatrix}\right]$ is the Gaussian polynomial, defined by

$$\begin{bmatrix} n\\ m\end{bmatrix}=\frac{(q)_n}{(q)_m(q)_{n-m}}=\frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^m)(1-q)\cdots(1-q^{n-m})}$$

for $0 \leqslant m \leqslant n$, and $[{n \atop m}] = 0$ otherwise. (Note that (5.1) and (5.2) were generalized in a certain direction in [9], but we will not be concerned with that generalization.) Here we will prove an analogue of (5.1) for higher powers of $1 - q^m$ on the left-hand side of (5.1) (i.e., for $V_k(q)$ as in (4.3)). We begin with a finite analogue.

Theorem 4. *For $k \geqslant 1$ and $n \geqslant 1$, we have*

$$\sum_{m=1}^{n} \frac{(-1)^{m-1} q^{\binom{m+1}{2}+(k-1)m}}{(1-q^m)^k} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{j_1=1}^{n} \frac{q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{q^{j_k}}{1-q^{j_k}}. \tag{5.3}$$

Proof. We use induction on k and on n. For $k = 1$, (5.3) reduces to (5.2). Now we assume that (5.3) is true for some fixed $k \geqslant 1$. We want to show that it holds for $k + 1$, namely

$$\sum_{m=1}^{n} \frac{(-1)^{m-1} q^{\binom{m+1}{2}+km}}{(1-q^m)^{k+1}} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{j_0=1}^{n} \frac{q^{j_0}}{1-q^{j_0}} \sum_{j_1=1}^{j_0} \frac{q^{j_1}}{1-q^{j_1}} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{q^{j_k}}{1-q^{j_k}}. \tag{5.4}$$

This induction step, in turn, will be done by induction on n, following the proof in [6].

For $n = 1$, (5.4) holds true (both sides being equal to $(q/(1-q))^{k+1}$). We assume that (5.4) holds for $n - 1$. To show that it holds also for n, we must verify that the difference between (5.4) for n and (5.4) for $n - 1$ is a true identity. This difference is

$$\sum_{m=1}^{n} \frac{(-1)^{m-1} q^{\binom{m+1}{2}+km}}{(1-q^m)^{k+1}} \left\{ \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} \right\}$$
$$= \frac{q^n}{1-q^n} \sum_{j_1=1}^{n} \frac{q^{j_1}}{1-q^{j_1}} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{q^{j_k}}{1-q^{j_k}}. \tag{5.5}$$

Now we use the identities

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} = q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = q^{n-m} \frac{1-q^m}{1-q^n} \begin{bmatrix} n \\ m \end{bmatrix},$$

and we see immediately that (5.5) is equivalent to (5.3). Hence (5.4) holds for all $n \geqslant 1$, and consequently (5.3) holds for all $k \geqslant 1$ and all $n \geqslant 1$. $\square$

By taking the limit as $n \to \infty$, we get the following result.

Theorem 5. *For $k \geqslant 1$, we have*

$$\sum_{m=1}^{n} \frac{(-1)^{m-1} q^{\binom{m+1}{2}+(k-1)m}}{(q)_m (1-q^m)^k} = \sum_{j_1=1}^{\infty} \frac{q^{j_1}}{1-q^{j_1}} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{q^{j_k}}{1-q^{j_k}}. \tag{5.6}$$

Remark. Both (5.3) and (5.6) are true also for $k = 0$ if we interpret the right-hand sides to be 1. Eq. (5.3) for $k = 0$ is the Lemma in [6], and (5.6) for $k = 0$ follows by taking the limit as $n \to \infty$.

We finish this section by stating explicitly a generalization of Uchimura's [8] formula

$$\sum_{n=1}^{\infty} nq^n \prod_{j=n+1}^{\infty} (1 - q^j) = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

Corollary 2. *For $k \geqslant 1$, we have*

$$\sum_{n=k}^{\infty} \binom{n}{k} q^n \prod_{j=n+1}^{\infty} (1 - q^j) = \sum_{j_1=1}^{\infty} \frac{q^{j_1}}{1 - q^{j_1}} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{q^{j_k}}{1 - q^{j_k}}. \tag{5.7}$$

Proof. This follows directly from Theorem 5 and Lemma 1 if we note that

$$\binom{m+k}{2} - \binom{k}{2} = \binom{m+1}{2} + (k-1)m. \qquad \square$$

We remark that (6.7) holds for $k = 0$ as well. Indeed, the identity

$$\sum_{n=0}^{\infty} q^n \prod_{j=n+1}^{\infty} (1 - q^j) = 1$$

is a special case of the q-binomial identity.

6. Some binomial identities

As was pointed out in [3], the partial sums

$$H_n(q) := \sum_{j=1}^{n} \frac{q^j}{1 - q^j}$$

can be considered q-analogues of the harmonic numbers

$$H_n := \sum_{j=1}^{n} \frac{1}{j}.$$

Accordingly, the identity (5.2) is a q-analogue of the well-known trigonometric identity

$$\sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} \frac{1}{m} = \sum_{j=1}^{n} \frac{1}{j} \tag{6.1}$$

(see, e.g., [5, (1.46)]). To obtain a generalization of (6.1), we multiply both sides of (5.3) by $(1-q)^k$ and take the limit as $q \to 1$. The author was unable to find the following identity in the literature.

Corollary 3. *For* $k \geqslant 1$, *we have*

$$\sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} \frac{1}{m^k} = \sum \frac{1}{j_1 j_2 \cdots j_k},$$

where the sum extends over all integers $1 \leqslant j_k \leqslant j_{k-1} \leqslant \cdots \leqslant j_1 \leqslant n$.

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